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Large Deviation Approach to the Randomly Forced Navier–Stokes Equation

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The random forced Navier–Stokes equation can be obtained as a variational problem of a proper action. By virtue of incompressibility, the integration over transverse components of the fields allows to cast the action in the form of a large deviation functional. Since the hydrodynamic operator is nonlinear, the functional integral yielding the statistics of fluctuations can be practically computed by linearizing around a physical solution of the hydrodynamic equation. We show that this procedure yields the dimensional scaling predicted by K41 theory at the lowest perturbative order, where the perturbation parameter is the inverse Reynolds number. Moreover, an explicit expression of the prefactor of the scaling law is obtained.

KEY WORDS: Hydrodynamics; field theory; large fluctuations.

1. INTRODUCTION

A field theoretic approach to the study of the random stirred Navier–Stokes equation (rsNSE) can be traced back to the seminal paper by Martin, Siggia and Rose.[(1)] This was the starting point for the application of many field-theoretic strategies, e.g. diagramatic expansions, renormalization group methods[(2)] (for recent developments and applications the reader can be addressed to[(3)]), instanton-based approaches (for

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applications of instantonic methods in turbulence see, e.g.,[4–7] and references therein) and combinations of them.[8] The many technical difficulties encountered in developing these approaches avoided to gather conclusive achievements.

In this paper we show that one step forward along this field-theoretic approach allows one to cast the action associated with the rsNSE into the form of a large deviation functional. Recently, large-deviation theory has scored sensible success in describing fluctuations in stationary non-equilibrium regimes of various microscopic models.[9] This approach is mainly based on the extension of the time-reversal conjugacy property introduced by Onsager and Machlup[10] to stationary non-equilibrium states. In practice, thermal fluctuations in irreversible stationary processes can be traced back to a proper hydrodynamic description derived from the microscopic evolution rules. The general form of the action functional is

$$I_{[(t_1,t_2)]}(\rho)=\frac{1}{2}\int_{t_1}^{t_2} dt\ \langle W, K(\rho)\ W\rangle \tag{1}$$

where $\rho(t,\vec{x})$ represents in general a vector of thermodynamic variables depending on time t and space variables $\vec{x}$. The symbol $\langle\cdot\,,\,\cdot\rangle$ denotes the integration over space variables. W is a hydrodynamic evolution operator acting on ρ: it vanishes when ρ is equal to the stationary solution $\bar{\rho}$, which is assumed to be unique. The positive kernel $K(\rho)$ represents the stochasticity of the system at macroscopic level. According to the large deviation-theory, the entropy S of a stationary non-equilibrium state is related to the action functional I as follows:

$$S(\rho)=\inf_{\rho} I_{[-\infty,0]}(\hat{\rho}) \tag{2}$$

where the minimum is taken over all trajectories connecting $\bar{\rho}$ to $\hat{\rho}$.

For our purposes it is enough to consider that the action functional I provides a natural measure for statistical fluctuations in non-equilibrium stationary states, so that, formally, any statistical inference can be obtained from I. Indeed, from the very beginning we have to deal with a hydrodynamic formulation, namely the rsNSE: in the next section we will argue that an action functional of the form (1) can be obtained by field-theoretic analytic calculations.

In particular, explicit integration over all longitudinal components of the velocity field and over the associated auxiliary fields can be performed. This allows to obtain a hydrodynamic evolution operator W which depends only on the transverse components of the velocity field

$v_T^\alpha(t, \vec{x})$ $(\alpha = 1, 2, 3)$. Moreover, the positive kernel K amounts to the inverse correlation function of the stochastic source. This formulation allows to overcome some of the technical difficulties characterizing standard perturbative methods and diagramatic expansions.

On the other hand, we have to face with new difficulties. The hydrodynamic operator appearing in the large deviation functional is nonlinear, so that functional integration is unfeasible. One has to identify a solution $\bar{v}_T^\alpha(t, \vec{x})$ of the associated hydrodynamic equation and linearize the hydrodynamic operator around such a solution. Then, functional integration can be performed explicitly on the "fluctuation" field. In order to be well defined, this approximate procedure would demand the uniqueness of the solution of the nonlinear hydrodynamic equation. For this reason we have restricted our choice to a class of space–time functions which are also solutions of the linear problem. Among them, there is only one function which satisfies physically relevant boundary conditions (see Section 3). Statistical fluctuations have been estimated with respect to this solution, which has also the advantage of reducing the dependence of the generating functional on the pressure field to a trivial constraint. In practice, we construct a perturbative saddle-point approach based on a linearization procedure of the velocity field $v_T^\alpha(t, \vec{x})$ around $\bar{v}_T^\alpha(t, \vec{x})$. As a consequence of the nonlinear character of the original problem. the fluctuation field $u_T^\alpha(t, \vec{x}) = v_T^\alpha(t, \vec{x}) - \bar{v}_T^\alpha(t, \vec{x})$ is found to obey a linearized hydrodynamic problem with coefficients depending on space and time through $\bar{v}_T^\alpha(t, \vec{x})$. It is worth stressing that even the solution of the linearized problem is nontrivial and it is found to depend naturally on a perturbative parameter $\mathcal{R}^{-1}$, the inverse of the Reynolds number. We exploit this property by constructing a further perturbation procedure to obtain an explicit expression for $u_T^\alpha(t, \vec{x})$ at different orders in $\mathcal{R}^{-1}$. These points are discussed in Section 4.

Since our main purpose here is the estimation of the structure function (see Section 5) as an average over the non-equilibrium measure induced by the action I, we have to assume that the perturbative expansion applies in a wide range of values of $\mathcal{R}$. In particular, we guess that it holds also for moderately large $\mathcal{R}$, since a statistical average of any observable cannot be valid for too large values of $\mathcal{R}$, i.e. in a regime of fully developed turbulence. We will argue that statistical estimates can be consistently obtained for values of $\mathcal{R}$ which extend up to the region of stability of the solution $\bar{v}_T^\alpha(t, \vec{x})$. Beyond this region we have no practical way of controlling the convergence of the linearization procedure. It is worth stressing that we obtain an analytic expression of the structure function: the so-called K41 scaling law[11] is recovered on a spatial scale, whose nontrivial dependence on $\mathcal{R}$ is explicitly indicated.

At the present stage, we are not able to say at which extent our results on the dimensional scaling are dependent on the particular choice we did for the solution around which we studied the fluctuations. Further investigations are needed to clarify this important point, which probably require the combination of analytical and numerical techniques.

2. THE MODEL

We consider the Navier–Stokes equation for the velocity vector-field components $v^\alpha(t,\vec{x})$ describing a divergence-free homogeneous isotropic flow:

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)v^\alpha(t,\vec{x})+v^\beta(t,\vec{x})\frac{\partial}{\partial x^\beta}v^\alpha(t,\vec{x})+\frac{1}{\rho}\frac{\partial}{\partial x^\alpha}P(t,\vec{x})-f^\alpha(t,\vec{x})=0, \tag{3}$$

$$\frac{\partial}{\partial x^\alpha}v^\alpha(t,\vec{x})=0. \tag{4}$$

Here, P is the pressure and the field f^α represents a source/sink of momentum necessary to maintain velocity fluctuations. Customarily,[12] we assume f^α to be a white-in-time zero-mean Gaussian random force with covariance

$$\langle f^\alpha(t,\vec{x})f^\beta(t',\vec{x}')\rangle=F^{\alpha\beta}\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right). \tag{5}$$

Due to constraint (4), the field $v^\alpha(t,\vec{x})$ depends only on the transverse degrees of freedom of $f^\alpha(t,\vec{x})$. Without prejudice of generality we can also assume divergence-free forcing, yielding the additional relation

$$\frac{\partial}{\partial x^\alpha}F^{\alpha\beta}\left(\vec{x}-\vec{x}'\right)=\frac{\partial}{\partial x^\beta}F^{\alpha\beta}\left(\vec{x}-\vec{x}'\right)=0. \tag{6}$$

A standard choice for $F^{\alpha\beta}$ is

$$F^{\alpha\beta}(\vec{x})=\frac{D_0L^3}{(2\pi)^3}\int d^3p\ e^{i\vec{p}\cdot\vec{x}}(Lp)^s e^{-(Lp)^2}\mathcal{P}^{\alpha\beta}(p), \tag{7}$$

where D_0 is the power dissipated by the unitary mass, $p=|\vec{p}|$, L is the integral scale, s is an integer exponent (typically, $s=2$) and

$$\mathcal{P}^{\alpha\beta}(p)=\delta^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}$$

is the projector on the transverse degrees of freedom.

Following the Martin–Siggia–Rose formalism[1] we introduce the Navier–Stokes density of Lagrangian

$$\mathcal{L}(v, w, P, Q, f) = w^{\alpha}(t,\vec{x})\left[\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)v^{\alpha}(t,\vec{x}) + v^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}v^{\alpha}(t,\vec{x}) + \frac{1}{\rho}\frac{\partial}{\partial x^{\alpha}}P(t,\vec{x}) - f^{\alpha}(t,\vec{x})\right] + \frac{1}{\rho}Q(t,\vec{x})\frac{\partial}{\partial x^{\alpha}}v^{\alpha}(t,\vec{x}), \tag{8}$$

where the field w^{α} is the conjugated variable to the velocity field v^{α} and the field Q is the Lagrangian multiplier related to constraint (4). The generating functional is given by the integral

$$\mathcal{W}(J, P) = \int \mathcal{D}v\mathcal{D}w\mathcal{D}Q\mathcal{D}f \exp\left\{i\int dt\ d^3x\,[\mathcal{L}(v, w, P, Q, f) + J_{\alpha}v^{\alpha}] - \frac{1}{2}\int dt\, d^3x\, d^3y f^{\alpha} F^{-1}_{\alpha\beta} f^{\beta}\right\} \tag{9}$$

where J_{α} are the components of the "external source" vector J. By integration over the statistical measure, $\mathcal{D}f e^{-\frac{1}{2}\int fF^{-1}f}$ and over the Lagrange multiplier Q, we obtain an expression which depends only on the transverse component v_T of the velocity field v. By decomposing the auxiliary field w in terms of its transverse (w_T) and longitudinal (w_L) components, $w = w_L + w_T$, the measure $\mathcal{D}w$ factorizes into $\mathcal{D}w_L\mathcal{D}w_T$ and the generating functional (9) reduces to

$$\begin{aligned}\mathcal{W}(J, P) = \int \mathcal{D}w_T\mathcal{D}w_L\mathcal{D}v_T \exp &\\ \times\Bigg\{ i\int dt d^3x \Bigg[& w_T^{\alpha}\left\{\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)v_{\alpha T} + \left(v_T^{\beta}\frac{\partial}{\partial x^{\beta}}v_{\alpha T}\right)_T\right\} \\ & + w_L^{\alpha}\left\{\left(v_T^{\beta}\frac{\partial}{\partial x^{\beta}}v_{\alpha T}\right)_L + \frac{1}{\rho}\frac{\partial P}{\partial x^{\alpha}}\right\} + J_{\alpha}v_T^{\alpha}\Bigg] \\ & - \frac{1}{2}\int dt\int d^3x\, d^3y w_T^{\alpha}F_{\alpha\beta}w_T^{\beta}\Bigg\}.\end{aligned} \tag{10}$$

Diagramatic strategies are usually applied at this level. We want to point out that one can go further by observing that also the transverse and longitudinal components of the auxiliary field w can be integrated out, yielding the equation

$$\mathcal{W}(J,P)=\int \mathcal{D}v_T e^{-\frac{1}{2}I(v_T)+i\int dt d^3x J_\alpha v_T^\alpha}\delta\left(\left(v_T^\beta \frac{\partial}{\partial x^\beta}v_{\alpha T}\right)_L+\frac{1}{\rho}\frac{\partial P}{\partial x^\alpha}\right) \quad (11)$$

where the action functional I has the form

$$I(v_T)=\int dt d^3x d^3y \left[\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)v_T^\alpha(t,\vec{x})+v_T^\rho(t,\vec{x})\partial_\rho v_T^\alpha(t,\vec{x})\right]$$
$$F_{\alpha\beta}^{-1}(|\vec{x}-\vec{y}|)\left[\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)v_T^\beta(t,\vec{y})+v_T^\lambda(t,\vec{y})\partial_\lambda v_T^\beta(t,\vec{y})\right]. \quad (12)$$

The computation of (11) would require to solve the constraint

$$\left(v_T^\beta \frac{\partial}{\partial x^\beta}v_{\alpha T}\right)_L+\frac{1}{\rho}\frac{\partial P}{\partial x^\alpha}=0, \quad (13)$$

In principle, this is a very difficult task due to the nonlinear character ot the constraint.

In the following section we show that we can identify a particular extremal solution, $\bar{v}_T$, of the functional (12). This solution is found to be independent of the stochastic source and, moreover, it satisfies constraint (13) for any constant value of the pressure. Accordingly, $I(v_T)$ can be interpreted as a large deviation functional (see Eq. (1)) and the statistical nonequilibrium measure of the rsNSE can be effectively evaluated by integrating over the fluctuations around this extremal solution. It is worth observing that the entropy is related to the functional $I(v_T)$ by the relation[9]

$$S(v_T)=\frac{1}{2}\inf_{v_T} I(v_T), \quad (14)$$

where the minimum is taken over all trajectories connecting $\bar{v}_T$ to v_T.

In what follows we are going to show that a suitable perturbative strategy can be applied for obtaining explicit analytic calculations of the statistical properties of the rsNSE.

3. A QUASI-STEADY SOLUTION AND ITS STABILITY

Any analytic approach aiming at the estimation of the generating functional (11) demands the identification of an explicit solution of the action functional (12). In practice, this amounts to solve the stationarity condition

$$\frac{\delta I(v_T)}{\delta v_T^{\sigma}(t,\vec{x})} = 2\int d^3y\Big[-\delta_{\sigma}{}^{\alpha}\left(\frac{\partial}{\partial t}+\nu\nabla^2\right)+\partial_{\sigma}v_T^{\alpha}(t,\vec{x})$$
$$-\delta_{\sigma}{}^{\alpha}v_T^{\rho}(t,\vec{x})\partial_{\rho}\Big]\bigg(F_{\alpha\beta}^{-1}(|\vec{x}-\vec{y}|)\Big[\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)v_T^{\beta}(t,\vec{y})$$
$$+v_T^{\lambda}(t,\vec{y})\partial_{\lambda}v_T^{\beta}(t,\vec{y})\Big]\bigg)=0. \tag{15}$$

We want to observe that for any arbitrary scalar field $\Phi(t,\vec{x})$ a solution of the equation

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)v_T^{\beta}(t,\vec{x})+v_T^{\lambda}(t,\vec{x})\partial_{\lambda}v_T^{\beta}(t,\vec{x})=\partial^{\beta}\Phi(t,\vec{x}), \tag{16}$$

is also a solution of (15). Since $F^{\alpha\beta}(|\vec{x}-\vec{y}|)$ contains a projector on the transverse degrees of freedom we can fix, without prejudice of generality, the condition $\partial^{\beta}\Phi=0$. It is worth pointing out that, for what concerns Eq. (16), this condition implies also that the longitudinal component of the nonlinear term vanishes, i.e. the solution $v_T^{\lambda}(t,\vec{x})$ has to satisfy the additional condition

$$\partial_{\beta}\left(v_T^{\lambda}(t,\vec{x})\partial_{\lambda}v_T^{\beta}(t,\vec{x})\right)=0. \tag{17}$$

Several different solutions can be found: among them, the only one unaffected by divergences in space and time is the following:

$$\bar{v}_T^{\alpha}(t,\vec{x})=\frac{U^{\alpha}}{2}\left\{1+e^{-\frac{t}{\tau_D}}\sin\left(\frac{1}{2\sqrt{b^2-(\vec{a}\cdot\vec{b})^2}}\left(\vec{b}\wedge\vec{a}\right)\cdot\frac{\vec{x}}{L}\right)\right\},\quad \text{with } t>0. \tag{18}$$

The U^{α} are the components of the vector of velocity amplitude $\vec{U}$ $(U=|\vec{U}|)$, $\vec{a}=\frac{\vec{U}}{U}$ is the corresponding unit vector and $\vec{b}$ identifies a rotation axis. Both vectors $\vec{U}$ and $\vec{b}$ can be fixed arbitrarily. We assume also that the length-scale L is the same as the forcing integral scale defined in (7). This implies that solution (18) decays exponentially in time to the constant $\frac{U^{\alpha}}{2}$ with the rate $\tau_D=4L^2/\nu$, which is the diffusion time scale. The dependence of solution (18) on the Reynolds number $\mathcal{R}$ can be made explicit by the relation $\mathcal{R}=\frac{LU}{\nu}$, so that $\tau_D=4\nu\mathcal{R}^2/U^2$. Notice that condition (17) is trivially satisfied by solution (18), because

$$\bar{v}_T^{\beta}(t,\vec{x})\partial_\beta \bar{v}_T^{\alpha}(t,\vec{x})=0. \tag{19}$$

Accordinlgy, $\bar{v}_T^{\alpha}(t,\vec{x})$ is also a solution of the diffusion equation $\left(\partial_t-\nu\nabla^2\right)\bar{v}_T^{\alpha}(t,\vec{x})=0$. There are two main consequences to be pointed out: (i) as a solution of the linear diffusion equation $\bar{v}_T^{\alpha}(t,\vec{x})$ is unique, which is a crucial requirement for the large deviation approach; (ii) the solution has to be defined only for positive times.

Moreover, due to condition (19), the constraint (13) is trivially solved by $P=$constant.

For $|\vec{x}|\ll L$ solution (18) approximates a linear shear flow: this is well known to produce instabilities for sufficiently large Reynolds numbers $\mathcal{R}$. In this perspective, it is worth analyzing the dynamical stability of (18). For this aim we consider the perturbed velocity vector, whose components are:

$$v^{\alpha}(t,\vec{x})=\bar{v}_T^{\alpha}(t,\vec{x})+\delta v_T^{\alpha}(t,\vec{x}). \tag{20}$$

The perturbation vector δv_T^{α} is assumed to be much smaller than $\bar{v}_T^{\alpha}$ with respect to any proper functional measure μ, i.e. $|\delta v_T^{\alpha}(t,\vec{x})|_\mu \ll |v^{\alpha}(t,\vec{x})|_\mu, \forall t$ and $\forall \vec{x}$. One can substitute (20) into (16) with $\partial^\beta\Phi=0$, while assuming that it satisfies constraint (17). In the linear approximation one obtains an equation for $\delta v_T^{\alpha}(t,\vec{x})$, which can be solved explicitly by performing an expansion in the inverse Reynolds number $\mathcal{R}^{-1}$. As shown in Appendix A, the perturbation field vanishes and, accordingly, (18) is stable for sufficiently large times and Reynolds numbers and provided the following inequality holds:

$$\frac{8\nu^2\mathcal{R}}{U^2}k^2>1. \tag{21}$$

This inequality implies that for increasing values of $\mathcal{R}$ the band of unstable modes becomes thinner and thinner. As a consequence, solving the stability problem by expanding the solution of the linearized dynamics (A.1) in powers of $\mathcal{R}^{-1}$ is consistent with this finding. Since condition (21) has been derived by assuming $\mathcal{R}$ large, it is not in contradiction with the Landau scenario for the origin of turbulence.

In summary, $\bar{v}_T^{\alpha}(t,\vec{x})$ exhibits all the expected features of a physically relevant solution, which corresponds to stationarity conditions for the large-deviation functional. Accordingly, it can be effectively used for computing statistical non-equilibrium fluctuations of the rsNSE. In the next section we will exploit a saddle point strategy for performing explicit calculations from the generating functional.

4. PERTURBATIVE ANALYSIS OF THE GENERATING FUNCTIONAL

All statistical properties concerning the rsNSE are contained in the structure functions which can be obtained by performing derivatives of the generating functional (11) with respect to the current J^α. An explicit calculation is unfeasible due to the nonlinear character of the action functional $I(v_T)$. Since in the previous section we have identified the solution $\bar{v}_T^\alpha$, we can tackle the problem by introducing the velocity field $u_T^\alpha = v_T^\alpha - \bar{v}_T^\alpha$, which represents fluctuations with respect to $\bar{v}_T^\alpha$, and by applying a saddle-point strategy.

Due to the translational invariance of the functional measure, the generating functional (11) can be rewritten as

$$\mathcal{W}(J) = \int \mathcal{D}u_T \; e^{-\frac{1}{2}I(u_T)+i\int dt\, d^3x\, J_\alpha u_T^\alpha}. \tag{22}$$

A linearized expression for the action functional can be obtained by assuming that higher order terms in u_T^α generated by the saddle-point expansion around the solution $\bar{v}_T^\alpha$ are negligible with respect to the functional measure $\mathcal{D}u_T$:

$$\begin{aligned} I(u_T) = \int_0^\infty dt \int d^3x d^3y \Big[&(\partial_t - \nu\nabla_x^2) u_T^\alpha(\hat{x}) + \bar{v}_T^\rho(\hat{x})\partial_\rho u_T^\alpha(\hat{x}) \\ &+ u_T^\rho(\hat{x})\partial_\rho \bar{v}_T^\alpha(\hat{x}) \Big] F_{\alpha\beta}^{-1}(|\vec{x}-\vec{y}|) \\ &\times \Big[(\partial_t - \nu\nabla_y^2) u_T^\beta(\hat{y}) + \bar{v}_T^\lambda(\hat{y})\partial_\lambda u_T^\beta(\hat{y}) + u_T^\lambda(\hat{y})\partial_\lambda \bar{v}_T^\beta(\hat{y}) \Big]. \end{aligned} \tag{23}$$

We have also introduced the shorthand notation $\hat{x} \equiv (t, \vec{x})$.

Consistently with this perturbative approach, we can also assume that, at leading order, constraint (13) is still trivially solved by (18), i.e. the pressure P is a constant.

In this way the action functional (23) has a bilinear form in the field u_T^α, with coefficients depending on $\bar{v}_T^\alpha$. In order to perform explicit Gaussian integration of the generating functional one has first to understand how the technical difficulties inherent such a dependence can be circumvented. The first problem that we have to face with is that, since (18) is defined only for $t>0$, also (23) is defined for positive times. As we discuss in Appendix B, a standard procedure allows one to get rid of any singularity of the action integral that might emerge for $t \to 0^+$. This is a consequence of the stucture of the linearized hydrodynamic operator appearing in (23). The second problem concerns the possibility of obtaining an analytic expression for the generating functional. To this aim one can exploit a perturbative expansion of (18) in powers of the inverse Reynolds number $\mathcal{R}^{-1}$. Actually, it is worth rewriting the solution (18) making explicit its dependence on the Reynolds number:

$$\bar{v}_T^{\alpha}(t,\vec{x}) = \frac{U^{\alpha}}{2}\left\{1 + e^{-\frac{U^2}{4\nu\mathcal{R}^2}t}\sin\left(\frac{2}{\sqrt{b^2-(\vec{a}\cdot\vec{b})^2}}\frac{\left(\vec{b}\wedge\vec{U}\right)\cdot\vec{x}}{4\nu\mathcal{R}}\right)\right\} \quad \text{with } t>0. \tag{24}$$

Using $\mathcal{R}^{-1}$ as a perturbative parameter, one can expand $\bar{v}_T^{\alpha}$ at all orders in $\mathcal{R}^{-1}$. When this expansion is substituted into (23) at leading order the action functional, in Fourier transformed variables, takes the form

$$I(u_T) = \int \frac{d^4p}{(2\pi)^4} u_T^{\rho}(-\hat{p}) M_{\rho}{}^{\alpha}(-\hat{p}) F_{\alpha\beta}^{-1}(p) M^{\beta}{}_{\zeta}(\hat{p}) u_T^{\zeta}(\hat{p}) + O\left(\frac{1}{\mathcal{R}^2}\right). \tag{25}$$

We denote with $u_T^{\zeta}(\hat{p})$ the Fourier transform of the field $u_T^{\zeta}(\hat{x})$ with $\hat{p} \equiv (p_0, \vec{p})$, p_0 and $\vec{p}$ being the Fourier-conjugated variables of t and $\vec{x}$, respectively. We introduce the representation of the action functional in terms of the Fourier-transformed variables because this makes more transparent the diagonalization procedure required to arrive at the final result.

The hydrodynamic evolution term $M^{\beta}{}_{\zeta}(\hat{p}) u_T^{\zeta}(\hat{p})$ is given by the expression

$$M^{\beta}{}_{\zeta}(\hat{p}) u_T^{\zeta}(\hat{p}) = \left\{\delta^{\beta}{}_{\zeta}\left[i\left(p_0 + \frac{1}{2}\vec{p}\cdot\vec{U}\right) + \nu p^2 - \frac{C}{4}\vec{p}\cdot\vec{U}\frac{\left(\vec{b}\wedge\vec{U}\right)^{\gamma}}{4\nu\mathcal{R}}\partial_{p_{\gamma}}\right] - \frac{C}{4}U^{\beta}\frac{\left(\vec{b}\wedge\vec{U}\right)_{\zeta}}{4\nu\mathcal{R}}\right\} u_T^{\zeta}(\hat{p}), \tag{26}$$

where $C = \frac{2}{\sqrt{b^2-(\vec{a}\cdot\vec{b})^2}}$.

The next step in this calculation requires the diagonalization of the matrix ${M_{\rho}}^{\alpha}(-\hat{p})F^{-1\,\alpha\beta}(p){M^{\beta}}_{\zeta}(\hat{p})$. Since by definition the factor $[F^{\alpha\beta}(p)]^{-1}$ is proportional to the identity operator in the space of the transverse solutions[6] we have just to diagonalize the matrix of the hydrodynamic operator ${M^{\beta}}_{\zeta}(\hat{p})$.

The computation of the eigenvalues, λ_1, λ_2 and λ_3 of ${M^{\beta}}_{\zeta}(\hat{p})$ deserves lengthy calculations sketched in Appendix C. Hereafter, we report the final form of the generating functional:

$$\mathcal{W}(\eta)=\int \mathcal{J}(H)\mathcal{D}\phi_T\;\, e^{-\frac{1}{2}\int_{\hat{p}}\phi^{\rho}_T(-\hat{p})F^{-1}(p)I_{\rho\gamma}(\hat{p})\phi^{\gamma}_T(\hat{p})+i\int_{\hat{p}}\eta_{T\alpha}(-\hat{p})\phi^{\alpha}_T(\hat{p})}, \quad (27)$$

where we have used the short-hand notation $\int_{\hat{p}}\equiv\int\frac{d^4p}{(2\pi)^4}$ and

$$I_{\rho\gamma}(p)=\begin{pmatrix}\lambda_1^*(\hat{p})\lambda_1(\hat{p}) & 0 & 0\\ 0 & \lambda_2^*(\hat{p})\lambda_2(\hat{p}) & 0\\ 0 & 0 & \lambda_3^*(\hat{p})\lambda_3(\hat{p})\end{pmatrix}, \quad (28)$$

$\mathcal{J}(H)$ is the Jacobian of the basis transformation $u\longrightarrow\phi$, $J\longrightarrow\eta$ engendered by the matrix H, which diagonalizes ${M_{\rho}}^{\alpha}(\hat{p})$. It is worth pointing out that the transformed vector $\phi^{\alpha}_T(\hat{p})$ still represents transverse components. Gaussian integration yields the following expression of the *normalized* functional in terms of the η^{α} source fields

$$\mathcal{W}(\eta)=e^{-\frac{1}{2}\int_{\hat{p}}\eta^{\rho}_T(-\hat{p})F(p)I^{-1}_{\rho\gamma}\eta^{\gamma}_T(\hat{p})}. \quad (29)$$

In practice, the explicit computation of the structure functions can be accomplished by returning to the original representation, where the generating functional has the form

$$\mathcal{W}(J)=e^{-\frac{1}{2}\int_{\hat{p}}J^{\rho}_T(-\hat{p})F(p)\left(HI^{-1}H^T\right)_{\rho\sigma}(\hat{p})J^{\sigma}_T(\hat{p})}. \quad (30)$$

In the next section we are going to derive an explicit expression for the second-order structure function.

[6]More explicitly we have $[F^{\alpha\beta}(p)]^{-1}=F^{-1}(p)\mathcal{P}^{\alpha\beta}(p)$ where $F(p)=D_0L^3(Lp)^se^{-(Lp)^2}$.

5. SHORT-DISTANCE BEHAVIOUR OF THE SECOND ORDER STRUCTURE FUNCTION

The analytic expression obtained for the generating functional (30) allows one to obtain all the statistical information about the fluctuations around the basic solution $\bar{v}_T^\alpha$. In this section we perform the explicit calculation of the second-order structure function of the velocity field u^α, defined as

$$\begin{aligned} S_2 &= \langle |u_T(t,\vec{r}+\vec{x}) - u_T(t,\vec{x})|^2 \rangle \\ &= \langle \left| (u_T^\alpha(t,\vec{r}+\vec{x}) - u_T^\alpha(t,\vec{x}))(u_{T\alpha}(t,\vec{r}+\vec{x}) - u_{T\alpha}(t,\vec{x})) \right| \rangle, \end{aligned} \tag{31}$$

The brackets denote averages over the stochastic forcing.

By assuming isotropy and homogeneity of the velocity field u^α, expression (31) is expected to assume the typical form of a scale invariant function

$$S_2(r) = r^{\zeta_2} F_2\left(t, \frac{r}{L}\right). \tag{32}$$

Here $r = |\vec{r}|$ and L is the integral scale associated with the noise source. It is worth stressing that, at variance with fully developed turbulent regimes, here the assumption of isotropy and homogeneity have to be taken as a plausible hypothesis allowing for analytic computations.

We want to point out that any exponent ζ_n must be independent of the basis chosen for representing the functional $\mathcal{W}$. For the sake of simplicity, it is worth using (29) rather than (30) to obtain:

$$\begin{aligned} S_2(r) = &\left(\frac{\delta}{i\delta\eta_T^\alpha(t,\vec{x}+\vec{r})} - \frac{\delta}{i\delta\eta_T^\alpha(t,\vec{x})} \right) \\ &\times \left(\frac{\delta}{i\delta\eta_{T\alpha}(t,\vec{x}+\vec{r})} - \frac{\delta}{i\delta\eta_{T\alpha}(t,\vec{x})} \right) \mathcal{W}(\eta) \bigg|_{\eta=0}. \end{aligned} \tag{33}$$

As shown in Appendix D, it turns out that $S_2(r)$ can be rewritten as follows:

$$S_2(r) = -\frac{1}{\nu} \left(I_1(r) + I_2(r) \right). \tag{34}$$

where

$$I_1(r) = \frac{D_0}{(2\pi)^2} r^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma\left(\frac{s+3+2n}{2}\right)}{\Gamma(2n+4)} \left(\frac{r}{L}\right)^{2n} \tag{35}$$

and

$$I_2(r) = D_0 L^3 \frac{32\nu^2\mathcal{R}}{U^2}\left\{\int_0^\infty \frac{p^2 dp}{(2\pi)^2}(Lp)^s e^{-(Lp)^2}\int_{-1}^1 dx\left(e^{iprx}-1\right)\right.$$
$$\times\left(\sum_{l=1,2}\frac{\left(1-x^2\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}}\frac{\left[\sum_{m=0}^2 s_{lm}F_m\left(x,\frac{8\nu^2\mathcal{R}}{U^2}p^2;\Sigma,\Xi\right)+\frac{1}{2}\Sigma\frac{x^{\frac{2}{3}}}{(1-x^2)^{\frac{1}{3}}}\right]}{\prod_{i\neq l}\left(\sum_{k=0}^2\left(s_{lk}-s_{ik}\right)F_k\left(x,\frac{8\nu^2\mathcal{R}}{U^2}p^2;\Sigma,\Xi\right)\right)}\right.$$
$$\left.\left.+O\left(\frac{1}{\mathcal{R}^2}\right)\right)\right\}. \tag{36}$$

The coefficients s_{ij} and the functions F_i, together with their arguments, are specified in Appendix D.

The main contribution of the stochastic measure $p^{2+s}e^{-(Lp)^2}dp$ to the first integral in (36) comes from a narrow region of wavenumbers close to $\bar{p}$, where the function $p^{2+s}e^{-(Lp)^2}$ has its maximum, i.e.

$$\bar{p}=\frac{1}{L}\sqrt{\frac{s+2}{2}}. \tag{37}$$

Accordingly, the function $\frac{8\nu^2\mathcal{R}}{U^2}p^2$ contributes to the integral by taking values close to $\frac{4(s+2)}{\mathcal{R}}$.

Moreover, for $p=\bar{p}$ the sufficient condition (21) for the stability of small perturbations determines an upper bound for the Reynolds number:

$$\mathcal{R}\lesssim 4(s+2). \tag{38}$$

This implies that for sufficiently small $\mathcal{R}$ the wavenumber $\bar{p}$ is stable. Under this condition, the leading contribution in (36), consistently with the expansion in $\mathcal{R}^{-1}$, can be obtained by performing an expansion in powers of $\frac{U^2}{8\nu^2\mathcal{R}p^2}$.

One finally obtains the complete expression of the structure function (see Appendix D for details)

$$S_2(r) = -\frac{1}{\nu}\left(I_1(r)+I_2(r)\right)\sim-\frac{D_0}{(2\pi)^2\nu}r^2\sum_{n=0}^{\infty}\left\{(-1)^{n+1}\Gamma\left(\frac{s+2n+3}{2}\right)\right.$$
$$\left.\times\left[\frac{1+\Xi}{\Gamma(2n+4)}-\frac{2^{\frac{13}{3}}\Xi}{\Sigma^{\frac{2}{3}}}\frac{2n+4}{\Gamma(2n+6)}\right]\left(\frac{r}{L}\right)^{2n}\right\}\quad \text{for } 1<\mathcal{R}\ll 4(2+s), \tag{39}$$

At leading order in the distance r this expression is dominated by a dissipative contribution.

We conjecture that this analysis can be extended to the parameter region defined by the condition $\mathcal{R} \gtrsim 4(2+s)$, where the statistically relevant wavenumbers can be unstable. As shown in Appendix D, in this case $I_2(r)$ has two contributions: one is again dissipative, while there is another one yielding the nontrivial scaling behavior $r^{2/3}$. Specifically, the expression of $S_2(r)$ for $\mathcal{R} \gtrsim 4(2+s)$ is found to be

$$S_2(r) \sim -\frac{D_0}{\pi\nu}\left\{\frac{1+\frac{\Xi}{2}}{4\pi}r^2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\Gamma\left(\frac{s+2n+3}{2}\right)}{\Gamma(2n+4)}\left(\frac{r}{L}\right)^{2n} + \frac{\mathcal{R}^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\left(\frac{\nu}{U}\right)^{\frac{4}{3}} r^{\frac{2}{3}}\sum_{n=0}^{\infty}C_n(\Sigma)\Gamma\left(\frac{3s+3n+5}{6}\right)\left(\frac{r}{L}\right)^{n}\right\}. \tag{40}$$

This expression is dominated by the term $r^{2/3}$ for sufficiently small distances. Indeed, the crossover scale between the r^2 and the $r^{\frac{2}{3}}$ terms occurs at

$$\frac{r}{L} \sim F\mathcal{R}^{-\frac{3}{4}}. \tag{41}$$

In Appendix D we evaluate the constant $F \sim 0.6$ and we report the expression of the numerical coefficient $C_0(\Sigma)$. The general expression of the coefficients $C_n(\Sigma)$ appearing in (40) has been omitted, because it has no practical interest for explicit calculations.

It is a remarkable fact that S_2 can exhibit the scaling behavior predicted by the K41 theory, which is assumed to hold (apart from intermittency corrections) when the velocity fluctuations are turbulent in the so-called inertial range of scales. This suggests that hydrodynamic fluctuations in a system at the very initial stage of instability development already contain some properties attributed to the developed turbulence regime.

6. CONCLUSIONS

In this paper we have exploited the field-theoretic approach to reformulate the random forced Navier–Stokes problem in terms of the evaluation of a quadratic action. This has the formal structure of a large-deviation functional, describing thermal fluctuations of irreversible stationary processes. The crucial step for obtaining such a statistical representation is the integration over all longitudinal components of both velocity and the associated auxiliary fields. With respect to the standard formulation which yields usual diagramatic strategies, we perform one more field integration. The positive definite kernel, which connects the hydrodynamic evolution operator in the action functional, is the inverse of the forcing correlation function.

In terms of the action functional, the knowledge of the whole velocity statistics reduces to the computation of functional integrals. However, due to the intrinsic nonlinear character of the hydrodynamic operator several technical difficulties have been solved for performing analytic calculations. In particular, one has to introduce suitable approximations.

In order to obtain an analytic expression of the generating functional we have identified a solution around which we have linearized the hydrodynamic evolution operator. We have also introduced a velocity field which represents fluctuations with respect to this solution. A perturbative expansion in the inverse Reynolds number finally yields the wanted result.

In principle, from this analytic treatment one can obtain all relevant statistical information about the rsNSE by computing any velocity multipoint structure function. In this paper we report only the explicit calculation of the two-point second-order moment of the velocity field. As shown in the Appendices, the algebraic manipulations needed for obtaining the final result are far from trivial also in this simple case.

In fact, in this paper we aim at understanding whether fluctuations at the early stage of their development (accordingly, we dub them as preturbulent fluctuations) already contain some important features of developed turbulence. We are interested, in particular, to characterize the scale invariant properties of such fluctuations. In this respect, we find that they are organized at different scales in a self-similar way. Remarkably, the scaling exponent coincides with the dimensional prediction of the Kolmogorov 1941 theory,[11] valid for developed turbulence regimes. Whether or not such exponent is a genuine reminiscence of the developed turbulence phenomenology needs further investigations.

Unfortunately, the complexity of the derivation leading to the K41 scaling law does not allow us to identify precisely the very origin of such a dimensional prediction. We can however argue a relationship between the observed dimensional scaling and the conservation laws (for momentum and energy) associated with the two eigenvalues of the matrix appearing in the action functional (25).

Finally, it is worth observing that the dimensional scaling law emerges for a particular choice we did for the pressure field: fluctautions have been restricted around a solution for which the pressure is constant. Unfortunately, owing to the fact that the analytical treatment is not *duable* in the general case, we cannot substantiate the fact on whether the dimensional prediction we found is not a consequance of our particular choice for the pressure fields.

At least three scenaries might be possible. Firstly, pressure field does not affect neither the leading (dimensional) scaling law nor its prefactor. It only affects the subleading scaling contributions. In this case our simplification would capture the relevant physics of the problem. The second possibility is that the leading scaling law does not change but this is not for the prefactor. The last possibility is that pressure changes the (dimensional) scaling law giving rise to intermittency corrections. Unfortunately, at the present stage of our knowledge, we are not in the position to select one scenary among the three we have pointed out. Further investigations are needed for this aim, which probably call to deep numerical investigations of the system under consideration.

We want to conclude by outlining some open problems and perspectives. A first question concerns the physical relevance of the solution (18) around which we linerize the evolution operator. It represents a shear-like solution, which is a well-known generator of instability. Moreover, its unicity and stability properties seem to indicate that this solution can play a major role in the determination of stationary nonequilibrium fluctuation statistics to be attributed to the rsNSE. As a mathematical object, it exhibits all the wanted features that one would like to attribute to such a solution. On the other hand, the authors have not yet a physical intuition for its relevance and aim at making some future progress in this direction.

Another interesting point to be tackled concerns the computation of the third-order momentum of the velocity correlators. In this case the predictions of our approach could be compared with the 4/5-law, which is one among the very few exact results of turbulence theories.

Finally, the extension of our results to other classes of transport problems, including passive scalar advection, could provide a better understanding of the basic mechanism at the origin of the observed scaling behaviors.

APPENDIX A

In this appendix we perform the stability analysis of the solution $\bar{v}_T^{\alpha}$ by the linearized equation

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\delta v_T^{\gamma}(t,\vec{x})+\bar{v}_T^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}\delta v_T^{\gamma}(t,\vec{x})+\delta v_T^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}\bar{v}_T^{\gamma}(t,\vec{x})=0 \tag{A.1}$$

with the constraint

$$\frac{\partial}{\partial x^{\gamma}}\left(\bar{v}_T^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}\delta v_T^{\gamma}(t,\vec{x})+\delta va_T^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}\bar{v}_T^{\gamma}(t,\vec{x})\right)=0.$$

In Section 3 we have already observed that $\bar{v}_T^{\alpha}$ is a quasi-steady solution for a time $t \ll \tau_D = \frac{4\nu\mathcal{R}^2}{U^2}$. The Fourier transform of Eq. (A.1) with respect to the space vector $\vec{x}$ yields:

$$\begin{aligned}&\frac{\partial}{\partial t}\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}\right)-\nu k^2\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}\right)+\frac{i}{2}\vec{k}\cdot\vec{U}\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}\right)\\&\quad+\frac{1}{4}e^{-\frac{t}{\tau_D}}\left\{U^{\beta}k_{\beta}\left[\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}-C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right)-\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}+C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right)\right]\right.\\&\quad\left.+U^{\alpha}C\frac{\left(\vec{b}\wedge\vec{U}\right)_{\beta}}{4\nu\mathcal{R}}\left[\delta\tilde{v}_T^{\beta}\left(t,\vec{k}-C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right)+\delta\tilde{v}_T^{\beta}\left(t,\vec{k}+C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right)\right]\right\}\\&=0,\end{aligned}\tag{A.2}$$

where $C=\frac{2}{\sqrt{b^2-(\vec{a}\cdot\vec{b})^2}}$. By performing a perturbative expansion up to second-order in the parameter $\mathcal{R}^{-1}$, one obtains the system of equations

$$\frac{\partial}{\partial t}\delta\tilde{v}_{T(0)}^{\alpha}\left(t,\vec{k}\right)+\nu k^2\delta\tilde{v}_{T(0)}^{\alpha}\left(t,\vec{k}\right)+\frac{i}{2}\vec{k}\cdot\vec{U}\delta\tilde{v}_{T(0)}^{\alpha}\left(t,\vec{k}\right)=0,\tag{A.3}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\delta\tilde{v}_{T(1)}^{\alpha}\left(t,\vec{k}\right)+\nu k^2\delta\tilde{v}_{T(1)}^{\alpha}\left(t,\vec{k}\right)+\frac{i}{2}\vec{k}\cdot\vec{U}\delta\tilde{v}_{T(1)}^{\alpha}\left(t,\vec{k}\right)\\&=\frac{1}{2}\vec{k}\cdot\vec{U}C\frac{\left(\vec{b}\wedge\vec{U}\right)_{\beta}}{4\nu\mathcal{R}}\frac{\partial}{\partial k_{\beta}}\delta\tilde{v}_{T(0)}^{\alpha}\left(t,\vec{k}\right)-\frac{1}{2}U^{\alpha}C\frac{\left(\vec{b}\wedge\vec{U}\right)_{\beta}}{4\nu\mathcal{R}}\delta\tilde{v}_{T(0)}^{\beta}\left(t,\vec{k}\right),\end{aligned}\tag{A.4}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\delta\tilde{v}_{T(2)}^{\alpha}\left(t,\vec{k}\right)+\nu k^2\delta\tilde{v}_{T(2)}^{\alpha}\left(t,\vec{k}\right)+\frac{i}{2}\vec{k}\cdot\vec{U}\delta\tilde{v}_{T(2)}^{\alpha}\left(t,\vec{k}\right)\\&=\frac{1}{2}\vec{k}\cdot\vec{U}C\frac{\left(\vec{b}\wedge\vec{U}\right)_{\beta}}{4\nu\mathcal{R}}\frac{\partial}{\partial k_{\beta}}\delta\tilde{v}_{T(1)}^{\alpha}\left(t,\vec{k}\right)-\frac{1}{2}U^{\alpha}C\frac{\left(\vec{b}\wedge\vec{U}\right)_{\beta}}{4\nu\mathcal{R}}\delta\tilde{v}_{T(1)}^{\beta}\left(t,\vec{k}\right).\end{aligned}\tag{A.5}$$

This system of equations yields the perturbative solution

$$\begin{aligned}\delta\tilde{v}_T^{\alpha}\left(t,\vec{k}\right) &= e^{-\left(\nu k^2+\frac{i}{2}\vec{U}\cdot\vec{k}\right)t}\Bigg\{F_{(0)}^{\alpha}\left(\vec{k}\right)+F_{(1)}^{\alpha}\left(\vec{k}\right)\\ &\quad +C\frac{\vec{k}\cdot\vec{U}}{8\nu\mathcal{R}}\Bigg[\left(\vec{b}\wedge\vec{U}\right)\cdot\vec{\nabla}_k F_{(0)}^{\alpha}\left(\vec{k}\right)t-\frac{U^{\alpha}}{\vec{k}\cdot\vec{U}}\left(\vec{b}\wedge\vec{U}\right)\cdot\vec{F}_{(0)}\left(\vec{k}\right)t\\ &\quad -\left(\vec{b}\wedge\vec{U}\right)\cdot\vec{k}F_{(0)}^{\alpha}\left(\vec{k}\right)\nu t^2\Bigg]+O\left(\frac{1}{\mathcal{R}^2}\right)\Bigg\} \qquad \text{(A.6)}\end{aligned}$$

where the functions Fs are determined by the initial conditions: they are found to be of $O(1)$ for any k.

The exponential term in front of (A.6) makes the perturbative solution vanish in the limit of large time t, provided the perturbative series contained in the curly brackets does not diverge faster in such a limit. This requirement can be translated into the following spectral condition:

$$\frac{8\nu^2\mathcal{R}}{U^2}k^2>1. \qquad \text{(A.7)}$$

This inequality indicates that the instability of solution (18) may originate only from sufficiently small values of the wavenumber k.

APPENDIX B

As shown in Section 3 the solution $\bar{v}_T^{\alpha}$ of the hydrodynamic operator in the action functional (12) is defined for $t>0$. Accordingly, it breaks Galilean invariance, thus giving rise to the well-known Doppler effect, i.e. $k_0\to k_0+\frac{1}{2}\vec{k}\cdot\vec{U}$.

Moreover, since in Section 4 we evaluate the action functional by applying a saddle-point expansion around $\bar{v}_T^{\alpha}$, the approximated expression (23) contains a time integral that has to be restricted to $t>0$ only. This amounts to assume that the action should be identically zero for $t<0$. Accordingly, one cannot exclude the possibility that a singularity in the time integral may originate at $t=0$.

In this appendix we want to show that one can easily exclude the presence of any singularity by passing to a Fourier-transformed representation of the action functional (23): according to a standard field-theoretic technique the addition of a small immaginary part to the frequency appearing in the Fourier-transformed integral allows one to control its regular behavior for $t\to 0^+$.

For the sake of clarity, we present this procedure only for two of the terms appearing in (23). Actually, one can easily realize that the procedure can be extended to all the terms: we just report the final result, thus avoiding the writing of lengthy formulae.

Let us consider the term

$$I_1 = \int_0^{\infty} dt \int d^3x \int d^3y \frac{\partial}{\partial t} u_T^{\alpha}(t,\vec{x}) F^{-1\alpha\beta}(|\vec{x}-\vec{y}|) \frac{\partial}{\partial t} u_T^{\beta}(t,\vec{y}). \tag{B.1}$$

In principle, the integral in the time domain is ill-defined. We can pass to Fourier-transformed variables and rewrite it as follows:

$$I_1 = -\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \int_0^{+\infty} dt\, e^{i(k_0+q_0)t} \tilde{u}_T^{\alpha}(k_0,\vec{k}) \frac{k_0 q_0}{F^{\alpha\beta}(k)} \tilde{u}_T^{\beta}(q_0,-\vec{k}). \tag{B.2}$$

The time integral can be regularized by adding a small immaginary part $i\epsilon$ to the frequency component and the integral I_1 is transformed into

$$\begin{aligned} I_1' &= -\int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int_0^{+\infty} dt\, e^{i(k_0+q_0+i\epsilon)t} \tilde{u}_T^{\alpha}(k_0,\vec{k}) k_0 q_0 \tilde{u}_T^{\beta}(q_0,-\vec{k}) \\ &= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \frac{k_0 q_0}{i(k_0+q_0+i\epsilon)} \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(q_0,-\vec{k}) \\ &= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \frac{k_0(q_0-k_0)}{i(q_0+i\epsilon)} \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(q_0-k_0,-\vec{k}). \end{aligned} \tag{B.3}$$

By performing the limit $\epsilon \to 0^+$ one obtains

$$\begin{aligned} I_1' &= -\int \frac{d^3k}{(2\pi)^3} i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \left[P \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \frac{1}{q_0} k_0(q_0-k_0) \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(q_0-k_0,-\vec{k}) \right. \\ &\quad \left. -i\pi \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \delta(q_0) k_0(q_0-k_0) \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(q_0-k_0,-\vec{k}) \right] \\ &= -i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \left[\frac{1}{2\pi} P \int_{-\infty}^{+\infty} dq_0 \frac{k_0(q_0-k_0)}{q_0} \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(q_0-k_0,-\vec{k}) \right. \\ &\quad \left. + \frac{i}{2} k_0^2 \tilde{u}_T^{\alpha}(k_0,\vec{k}) \tilde{u}_T^{\beta}(-k_0,-\vec{k}) \right]. \end{aligned} \tag{B.4}$$

In this equation P denotes the principal value. The nontrivial part to be computed is contained in the square brackets. One has to consider that the fluctuations $u_T^{\alpha}(t,\vec{x})$ become negligible for scales smaller than the Kolmogorov scale. Since they are defined for $t>0$ and the time integral is singular in $t=0$, we have that its Fourier-transformed representation should exhibit a unique singularity at infinity, where it vanishes for $Im\ q_0 < 0$. One can write:

$$\begin{aligned}
&\frac{1}{2\pi} P \int_{-\infty}^{+\infty} dq_0 \frac{k_0(q_0-k_0)}{q_0} \tilde{u}_T^\alpha(k_0,\vec{k})\tilde{u}_T^\beta(q_0-k_0,-\vec{k}) \\
&= -\frac{k_0^2\tilde{u}_T^\alpha(k_0,\vec{k})}{2\pi} P \int_{-\infty}^{+\infty} dq_0 \frac{\tilde{u}_T^\beta(q_0-k_0,-\vec{k})}{q_0} \\
&= \frac{i}{2} k_0^2 \tilde{u}_T^\alpha(k_0,\vec{k})\tilde{u}_T^\beta(-k_0,-\vec{k}).
\end{aligned} \tag{B.5}$$

Making use of this result, one can easily conclude that (B.2) can be written as follows:

$$I_1 = \int \frac{dk_0 d^3k}{(2\pi)^4} k_0^2 \tilde{u}_T^\alpha(k_0,\vec{k}) F^{-1\alpha\beta}(k)\tilde{u}_T^\beta(-k_0,-\vec{k}). \tag{B.6}$$

Now, let us consider one of the terms of (23) which exhibits the Doppler effect in its Fourier-transformed representation:

$$\begin{aligned}
I_2 &= \int_0^\infty dt \int d^3x \int d^3y \frac{\partial}{\partial t} u_T^\alpha(t,\vec{x}) F^{-1\alpha\beta}(|\vec{x}-\vec{y}|)\bar{v}_T^\lambda(t,\vec{y})\partial_\lambda u_T^\beta(t,\vec{y}) \\
&= \frac{U^\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \left\{ \int_0^{+\infty} dt\ e^{i(k_0+q_0+i\epsilon)t} \tilde{u}_T^\alpha(k_0,\vec{k}) \frac{k_0 k_\lambda}{F^{\alpha\beta}(k)} \tilde{u}_T^\beta(q_0,-\vec{k}) \right. \\
&\quad + \int_0^{+\infty} dt\, e^{i(k_0+q_0+i\frac{U^2}{4\nu\mathcal{R}^2})t} \frac{i}{2} \tilde{u}_T^\alpha(k_0,\vec{k}) \frac{k_0}{F^{\alpha\beta}(k)} \left[-\left(k_\lambda + C\frac{(\vec{b}\wedge\vec{U})_\lambda}{4\nu\mathcal{R}}\right) \tilde{u}_T^\beta\left(q_0, -\vec{k} - C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right) \right. \\
&\quad \left.\left. + \left(k_\lambda - C\frac{(\vec{b}\wedge\vec{U})_\lambda}{4\nu\mathcal{R}}\right) \tilde{u}_T^\beta\left(q_0, -\vec{k} + C\frac{\vec{b}\wedge\vec{U}}{4\nu\mathcal{R}}\right) \right] \right\} \\
&= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \left\{ \frac{i}{k_0+q_0+i\epsilon} \tilde{u}_T^\alpha(k_0,\vec{k}) \frac{k_0(\vec{k}\cdot\vec{U})}{F^{\alpha\beta}(k)} \tilde{u}_T^\beta(q_0,-\vec{k}) \right. \\
&\quad \left. - \frac{1}{(k_0+q_0+i\frac{U^2}{4\nu\mathcal{R}^2})} \tilde{u}_T^\alpha(k_0,\vec{k}) \frac{k_0(\vec{k}\cdot\vec{U})}{F^{\alpha\beta}(k)} C \frac{(\vec{b}\wedge\vec{U})_\lambda}{4\nu\mathcal{R}} \frac{\partial}{\partial k_\lambda} \tilde{u}_T^\beta(q_0,-\vec{k}) + O\left(\frac{1}{\mathcal{R}^2}\right) \right\}.
\end{aligned} \tag{B.7}$$

We expand the solution $\bar{v}_T^\lambda$ up to first order in powers of $\mathcal{R}^{-1}$ and we obtain the final expression:

$$\begin{aligned}
I_2 &= \frac{1}{2} \int \frac{dk_0\, d^3k}{(2\pi)^4} \tilde{u}_T^\alpha(k_0,\vec{k}) \frac{k_0(\vec{k}\cdot\vec{U})}{F^{\alpha\beta}(k)} \\
&\quad \times \left\{ \tilde{u}_T^\beta(-k_0,-\vec{k}) + iC\frac{(\vec{b}\wedge\vec{U})_\lambda}{8\nu\mathcal{R}} \frac{\partial}{\partial k_\lambda} \tilde{u}_T^\beta(-k_0,-\vec{k}) + O\left(\frac{1}{\mathcal{R}^2}\right) \right\}.
\end{aligned}$$

As in the previous case, one can regularize the integral in $t=0$ by performing the limit $\epsilon \to 0^+$. By applying this procedure to all of the remaining terms in (23) one arrives at the final expression (25).

APPENDIX C

In this appendix we sketch the calculation of the eigenvalues of the matrix $M_\zeta^\beta(\hat{p})$ defined in (26). In fact, the perturbative expansion of the solution (18) in powers of $\frac{1}{\mathcal{R}}$ induces an analogous expansion for this matrix. Formally, one can write

$$M = M_{(0)} + M_{(1)} + \cdots , \tag{C.1}$$

where

$$\begin{aligned} M^\alpha_{(0)\ \beta} &= \delta^\alpha{}_\beta \left[i\left(p_0 + \frac{1}{2}\vec{p}\cdot\vec{U}\right) + \nu p^2 \right], \\ M^\alpha_{(1)\ \beta} &= -\delta^\alpha{}_\beta \frac{C}{4}\vec{p}\cdot\vec{U}\frac{\left(\vec{b}\wedge\vec{U}\right)^\gamma}{4\nu\mathcal{R}}\partial_{p_\gamma} - \frac{C}{4}U^\alpha \frac{\left(\vec{b}\wedge\vec{U}\right)_\beta}{4\nu\mathcal{R}}. \end{aligned} \tag{C.2}$$

The matrix $M_\zeta^\beta(\hat{p})$ acts on the two-dimensional space of the transverse functions and on the one-dimensional space of the longitudinal functions. Only the transverse degrees of freedom are physically relevant.

A complete orthonormal basis in R^3 is given by the vectors

$$\begin{aligned} \Pi_1^\alpha &= \frac{\left(\vec{b}\wedge\vec{p}\right)^\alpha}{\sqrt{f(p)}}, \\ \Pi_2^\alpha &= \frac{g(p)\left(\vec{b}\wedge\vec{p}\right)^\alpha - f(p)\left(\vec{U}\wedge\vec{p}\right)^\alpha}{\sqrt{f(p)}\sqrt{f(p)h(p)-g^2(p)}}, \\ \Pi_3^\alpha &= \frac{p^\alpha}{p}, \end{aligned} \tag{C.3}$$

where

$$\begin{gathered} f(p) = b^2p^2 - (\vec{b}\cdot\vec{p})^2, \quad g(p) = (\vec{b}\cdot\vec{U})p^2 - (\vec{b}\cdot\vec{p})(\vec{U}\cdot\vec{p}), \\ h(p) = U^2p^2 - (\vec{U}\cdot\vec{p})^2. \end{gathered} \tag{C.4}$$

Π_1^α and Π_2^α span the transverse subspace, while Π_3^α spans the longitudinal one. In analogy with (C.1), also the eigenvalues of $M_\zeta^\beta(\hat{p})$ can be represented by a perturbative expansion in powers of $\frac{1}{\mathcal{R}}$, namely as

$$\lambda^a = \lambda^a_{(0)} + \lambda^a_{(1)} + \cdots \quad \text{where} \quad a = 1, 2, 3. \tag{C.5}$$

The zero-order eigenvalues $\lambda^a_{(0)}$ are degenerate and have the form

$$\lambda^a_{(0)} = \left(i \left(p_0 + \frac{1}{2} \vec{p} \cdot \vec{U} \right) + \nu p^2 \right). \tag{C.6}$$

The evaluation of the first-order corrections $\lambda^a_{(1)}$ requires the diagonalization of the matrix with elements $M_{(1)ij} = \left(\Pi_i, M_{(1)} \Pi_j\right)$, $(i, j = 1, 2, 3)$. After some simple but lengthy calculations one finds

$$\begin{aligned}
\lambda^1_{(1)} &= \frac{1}{2} \left(M_{(1)11} + M_{(1)22} - \sqrt{\left(M_{(1)11} + M_{(1)22}\right)^2 + 4 M_{(1)21} M_{(1)12}} \right), \\
\lambda^2_{(1)} &= \frac{1}{2} \left(M_{(1)11} + M_{(1)22} + \sqrt{\left(M_{(1)11} + M_{(1)22}\right)^2 + 4 M_{(1)21} M_{(1)12}} \right), \\
\lambda^3_{(1)} &= M_{(1)33}
\end{aligned} \tag{C.7}$$

with

$$\begin{aligned}
M_{(1)11} &= \frac{C}{16\nu\mathcal{R}} \frac{\left(\vec{b} \wedge \vec{U}\right) \cdot \vec{p}}{f(p)} w(p), \\
M_{(1)22} &= -\frac{C}{16\nu\mathcal{R}} \frac{\left(\left(\vec{b} \wedge \vec{U}\right) \cdot \vec{p}\right) \left(\vec{b} \cdot \vec{p}\right) g(p)}{f(p) \left(f(p) h(p) - g^2(p)\right)} \\
&\quad \left[\left(\vec{p} \cdot \vec{U}\right) w(p) + (\vec{b} \cdot \vec{U}) g(p) - U^2 f(p) \right], \\
M_{(1)12} &= -\frac{C}{16\nu\mathcal{R}} \frac{\left(\left(\vec{b} \wedge \vec{U}\right) \cdot \vec{p}\right) \left(\vec{b} \cdot \vec{p}\right)}{f(p) \sqrt{f(p) h(p) - g^2(p)}} \\
&\quad \left[\left(\vec{b} \cdot \vec{U}\right) g(p) + 2 \left(\vec{p} \cdot \vec{U}\right) w(p) - U^2 f(p) \right], \\
M_{(1)21} &= -\frac{C}{16\nu\mathcal{R}} \frac{\left(\left(\vec{b} \wedge \vec{U}\right) \cdot \vec{p}\right)}{f(p) \sqrt{f(p) h(p) - g^2(p)}} \left(\vec{b} \cdot \vec{U}\right) \left[\left(\vec{b} \cdot \vec{p}\right) g(p) - \left(\vec{p} \cdot \vec{U}\right) f(p) \right], \\
M_{(1)33} &= -\frac{C}{16\nu\mathcal{R}} \frac{\left(\vec{p} \cdot \vec{U}\right)}{p^2} \left(\left(\vec{b} \wedge \vec{U}\right) \cdot \vec{p}\right),
\end{aligned} \tag{C.8}$$

where we have introduced the further definition:

$$w(p) = b^2(\vec{p}\cdot\vec{U}) - (\vec{b}\cdot\vec{p})(\vec{b}\cdot\vec{U}). \tag{C.9}$$

Without prejudice of generality, we can specify the geometrical structure of the flow. For the sake of simplicity, we assume that the vector $\vec{r}$ (i.e. the Fourier-conjugated variable of $\vec{p}$) corresponds to the polar axis and that the vector $\vec{b}$ is orthogonal to both $\vec{r}$ and $\vec{U}$. With this assumption the two physically relevant first-order corrections to the eigenvalues are

$$\begin{aligned}\lambda^1_{(1)} &= 0,\\ \lambda^2_{(1)} &= \frac{U^2}{16\nu\mathcal{R}}\Big\{\sin\theta_U\cos\theta_U\left[\cos^2\phi_U + \cos\left(2(\phi_U-\phi)\right)\right]\sin^2\theta\\ &\quad + \cos^2\theta_U \sin 2\theta\cos(\phi_U-\phi)\Big\}.\end{aligned} \tag{C.10}$$

Since $\lambda^3_{(i)}$ is associated to the longitudinal part, it does not play any role in our calculations.

APPENDIX D

In this appendix we aim at reporting the main calculations needed for obtaining an explicit expression for (33). According to the perturbative approach discussed in detail in Appendix C, $S_2(r)$ can be written as follows:

$$S_2(r) \sim -2\int \frac{dp_0\, d^3p}{(2\pi)^4}\left(e^{i\vec{p}\cdot\vec{r}}-1\right)\sum_{\alpha=1}^{2}\frac{F(p)}{\left(p_0+\frac{1}{2}\vec{p}\cdot\vec{U}\right)^2+\left(\nu p^2+\lambda^\alpha_{(1)}(\vec{p},\vec{U},\vec{b})\right)^2}. \tag{D.1}$$

The eigenvalues $\lambda^\alpha_{(1)}$ which appear in this equation have been computed up to first order of the perturbative expansion in $\mathcal{R}^{-1}$. Notice that the sum is restricted to the first two eigenvalues ($\alpha = 1, 2$), which correspond to the transverse components of the velocity field. Actually, the third eigenvalue, corresponding to the longitudinal components of the velocity field, is ineffective for our calculations.

Explicit integration over p_0 yields

$$S_2(r) \sim -\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{r}}-1}{\nu}\sum_{\alpha=1}^{2}\frac{F(p)}{p^2+\frac{1}{\nu}\lambda^\alpha_{(1)}(\vec{p},\vec{U},\vec{b})+\cdots}. \tag{D.2}$$

With the particular choice performed in Appendix C for the geometrical structure of the flow, $S_2(r)$ can be expressed as the sum of two terms: the first one is associated with the null eigenvalue $\lambda^1_{(1)}$, while the second one depends on the nonzero eigenvalue $\lambda^2_{(1)}$. Namely,

$$S_2(r) = -\frac{1}{\nu}\left(I_1(r) + I_2(r)\right). \tag{D.3}$$

By considering the explicit expressions of the statistical function $F(p)$ and of the eigenvalues $\lambda^{\alpha}_{(1)}$ (see Eq. (C.10)), one has

$$I_1(r) = D_0 L^3 \int \frac{d^3 p}{(2\pi)^3}\left(e^{i\vec{p}\cdot\vec{r}} - 1\right)\frac{(Lp)^s e^{-(Lp)^2}}{p^2}, \tag{D.4}$$

$$I_2(r) = D_0 L^3 \int \frac{d^3 p}{(2\pi)^3}\frac{\left(e^{i\vec{p}\cdot\vec{r}} - 1\right)(Lp)^s e^{-(Lp)^2}}{p^2 + \frac{U^2}{16\nu^2\mathcal{R}}\left[2\sin\theta_U\cos\theta_U\sin^2\theta\cos^2\phi + \cos^2\theta_U\sin 2\theta\cos\phi\right]}. \tag{D.5}$$

In the r.h.s. of this equation we have also exploited translational invariance for applying the transformation $(\phi_U - \phi) \to -\phi$. The analytic calculation of (D.4) is obtained by a standard procedure:

$$\begin{aligned} I_1(r) &= D_0 L^3 \int \frac{d^3 p}{(2\pi)^3}\left(e^{i\vec{p}\cdot\vec{r}} - 1\right)\frac{(Lp)^s e^{-(Lp)^2}}{p^2} \\ &= \frac{D_0 L^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!(2n+1)}\left(\frac{r}{L}\right)^{2n}\int_0^{\infty} d\zeta\ \zeta^{s+2n} e^{-\zeta^2} \\ &= \frac{D_0}{(2\pi)^2} r^2 \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\Gamma\left(\frac{s+3+2n}{2}\right)}{\Gamma(2n+4)}\left(\frac{r}{L}\right)^{2n}. \end{aligned} \tag{D.6}$$

For what concerns $I_2(r)$, we first perform the integration over the variable ϕ, namely:

$$I_2(r) = D_0 L^3 \int_0^{\infty}\frac{p^2 dp}{(2\pi)^3}(Lp)^s e^{-(Lp)^2}\int_{-1}^{+1} d(\cos\theta)\left(e^{ipr\cos\theta} - 1\right) I_0, \tag{D.7}$$

where

$$\begin{aligned} I_0 &= \int_0^{2\pi}\frac{d\phi}{p^2 + \frac{U^2}{16\nu^2\mathcal{R}}\left[2\sin\theta_U\cos\theta_U\sin^2\theta\cos^2\phi + \cos^2\theta_U\sin 2\theta\cos\phi\right]} \\ &= -i\,\frac{32\nu^2\mathcal{R}}{U^2}\int_{\gamma}\frac{z\,dz}{az^4 + bz^3 + cz^2 + bz + a} \end{aligned} \tag{D.8}$$

with $z=e^{i\phi}$ and the integration is on the unit circle γ. The coefficients a, b, c are given by

$$a=\sin\theta_U\cos\theta_U\sin^2\theta, \quad b=2\cos^2\theta_U\sin\theta\cos\theta$$
$$c=\frac{32\nu^2\mathcal{R}}{U^2}p^2+2\sin\theta_U\cos\theta_U\sin^2\theta. \tag{D.9}$$

The evaluation of the integral (D.8) requires the knowledge of the root of a fourth-order algebrical equation. By exploiting the Euler method[14] we end up with the expression

$$\begin{aligned} z_i &= z_i\left(x, \frac{8\nu^2\mathcal{R}}{U^2}p^2; \Sigma, \Xi\right) \\ &= \frac{x^{\frac{1}{3}}}{\left(1-x^2\right)^{\frac{1}{6}}}\left[\sum_{l=0}^{2} s_{il}F_l\left(x, \frac{8\nu^2\mathcal{R}}{U^2}p^2; \Sigma, \Xi\right)+\frac{1}{2}\Sigma\frac{x^{\frac{2}{3}}}{\left(1-x^2\right)^{\frac{1}{3}}}\right] \quad i=1,2,3,4. \end{aligned}$$

The following definition has been adopted:

$$\begin{aligned} F_l &= F_l\left(x, \frac{8\nu^2\mathcal{R}}{U^2}p^2; \Sigma, \Xi\right) \\ &= \left\{\frac{\Sigma^{\frac{2}{3}}}{12}\left[\frac{81}{4}\Sigma^4\frac{x^4}{\left(1-x^2\right)^2}+\frac{81}{2}\Sigma^2\frac{x^2}{\left(1-x^2\right)}-90\right.\right. \\ &\quad -\frac{64}{\Sigma^2}\frac{1-x^2}{x^2}+\frac{8\nu^2\mathcal{R}}{U^2}p^2\left(189\frac{\Sigma^2}{\Xi}\frac{x^2}{\left(1-x^2\right)^2}+\frac{382}{\Xi\left(1-x^2\right)}-120\frac{\Sigma^2}{\Xi\ x^2}\right) \\ &\quad +\left(\frac{8\nu^2\mathcal{R}}{U^2}p^2\right)^2\left(\frac{504}{\Xi^2\left(1-x^2\right)^2}+\frac{47\ \Sigma^2}{\Xi^2\ x^2\left(1-x^2\right)}\right) \\ &\quad \left.+\left(\frac{8\nu^2\mathcal{R}}{U^2}p^2\right)^3\frac{32\ \Sigma^2}{\Xi^3\ x^2\left(1-x^2\right)^2}\right]^{\frac{1}{3}} \\ &\quad \times\left(\epsilon^l\left[1+(1-4\times 27\ h)^{\frac{1}{2}}\right]^{\frac{1}{3}}+\epsilon^{l-3}\left[1-(1-4\times 27\ h)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right)+\frac{1}{2}\frac{\Sigma^{\frac{4}{3}}x^{\frac{4}{3}}}{\left(1-x^2\right)^{\frac{4}{6}}} \\ &\quad \left.+\frac{1}{3}\frac{\left(1-x^2\right)^{\frac{1}{3}}}{\Sigma^{\frac{2}{3}}x^{\frac{2}{3}}}+\frac{8\nu^2\mathcal{R}}{U^2}p^2\frac{2}{3\Sigma^{\frac{2}{3}}\Xi\ x^{\frac{2}{3}}\left(1-x^2\right)^{\frac{4}{6}}}\right\}^{\frac{1}{2}}, \end{aligned} \tag{D.10}$$

with

$$x = \cos\theta, \quad \Xi = \sin\theta_U \cos\theta_U, \quad \Sigma = \cot\theta_U, \tag{D.11}$$

$$s_{il} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}. \tag{D.12}$$

Here ϵ is the cubic root of unit: $\epsilon = \frac{-1+i\sqrt{3}}{2}$. The explicit expression of the function h follows:

$$\begin{aligned} h =& \left[16 + 30\ \Sigma^2 \frac{x^2}{1-x^2} + \frac{111}{4}\Sigma^4 \frac{x^4}{(1-x^2)^2} + 16\frac{8\nu^2\mathcal{R}}{U^2}\frac{p^2}{\Xi}\frac{17\Sigma^2x^2+3(1-x^2)}{(1-x^2)^2}\right. \\ &\left. +48\left(\frac{8\nu^2\mathcal{R}}{U^2}\frac{p^2}{\Xi}\right)^2\frac{1}{(1-x^2)^2}\right]^3 \times \left[-128 + 81\ \Sigma^4\frac{x^4}{(1-x^2)^2} + \frac{81}{2}\Sigma^6\frac{x^6}{(1-x^2)^3}\right. \\ & -180\Sigma^2\frac{x^2}{1-x^2} + \frac{8\nu^2\mathcal{R}}{U^2}\frac{p^2}{\Xi}\left(378\frac{\Sigma^4x^4}{(1-x^2)^3} + 764\frac{\Sigma^2x^2}{(1-x^2)^2} - 240\frac{1}{1-x^2}\right) \\ & \left. +\left(\frac{8\nu^2\mathcal{R}}{U^2}\frac{p^2}{\Xi}\right)^2\left(1008\frac{\Sigma^2x^2}{(1-x^2)^3} + 94\frac{1}{(1-x^2)^2}\right) + 64\left(\frac{8\nu^2\mathcal{R}}{U^2}\frac{p^2}{\Xi}\right)^3\frac{1}{(1-x^2)^3}\right]^{-2}. \end{aligned} \tag{D.13}$$

Only the roots z_1 and z_2 are included into the unit circle, therefore (D.7) becomes

$$\begin{aligned} I_2(r) =& D_0L^3\frac{32\nu^2\mathcal{R}}{U^2}\int_0^\infty \frac{p^2dp}{(2\pi)^2}(Lp)^s e^{-(Lp)^2}\int_{-1}^1 dx\left(e^{iprx}-1\right) \\ & \times \sum_{l=1,2}\frac{\left(1-x^2\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}}\frac{\left[\sum_{m=0}^2 s_{lm}F_m\left(x,\ \frac{8\nu^2\mathcal{R}}{U^2}p^2;\ \Sigma,\ \Xi\right) + \frac{1}{2}\Sigma\frac{x^{\frac{2}{3}}}{(1-x^2)^{\frac{1}{3}}}\right]}{\prod_{i\neq l}\left(\sum_{k=0}^2(s_{lk}-s_{ik})F_k\left(x,\ \frac{8\nu^2\mathcal{R}}{U^2}p^2;\ \Sigma,\ \Xi\right)\right)}. \end{aligned} \tag{D.14}$$

As we have already observed in Section V, only the values of the variable p around $\bar{p} = \frac{1}{L}\sqrt{\frac{s+2}{2}}$ give a significant contribution to the integral in (D.14). We observe that $\frac{8\nu^2\mathcal{R}}{U^2}\bar{p}^2 \rightarrow \frac{4(s+2)}{\mathcal{R}}$ and the stability condition (21) imposes:

$$1 < \mathcal{R} < 4(s+2). \tag{D.15}$$

The evaluation of the leading terms is then possible by performing an expansion in the parameter $\frac{U^2}{8\nu^2\mathcal{R}}p^{-2} \to \frac{\mathcal{R}}{8}\zeta^{-2}$ that, by virtue of (D.15), is smaller than unit if $\zeta < \sqrt{\frac{s+2}{2}}$.

For $\zeta > \sqrt{\frac{s+2}{2}}$ the contribution to the integral rapidly vanishes. For $1 < \mathcal{R} \ll 4(s+2)$ we obtain

$$\bar{S}_2(r) = -\frac{1}{\nu}\left(I_1(r) + I_2(r)\right) \sim -\frac{D_0}{(2\pi)^2\nu}r^2 \times \sum_{n=0}^{\infty}\left\{(-1)^{n+1}\Gamma\left(\frac{s+2n+3}{2}\right)\left[\frac{1+\Xi}{\Gamma(2n+4)} - \frac{2^{\frac{13}{3}}\,\Xi}{\Sigma^{\frac{2}{3}}}\frac{2n+4}{\Gamma(2n+6)}\right]\left(\frac{r}{L}\right)^{2n}\right\}. \tag{D.16}$$

By extending the validity of our calculations to $\mathcal{R} > 4(s+2)$, we have $\frac{8\nu^2\mathcal{R}}{U^2}p^2 \to \frac{8}{\mathcal{R}}\zeta^2 < 1$ for $\zeta < \sqrt{\frac{s+2}{2}}$. As in the previous case, we expand (D.14) in power of the parameter $\frac{8}{\mathcal{R}}\zeta^2 < 1$ and we obtain:

$$\begin{aligned} I_2(r) \sim\ & \Xi D_0 L^2 \int_0^{\infty}\frac{d\zeta}{(2\pi)^2}\zeta^s e^{-\zeta^2}\int_{-1}^{1}dx\left(e^{i\zeta\frac{r}{L}x}-1\right)\left\{\frac{1+\frac{8}{\mathcal{R}}\zeta^2+\cdots}{2}\right.\\ & + \frac{8}{\mathcal{R}\Xi}\sum_{l=1,2}\frac{\left(1-x^2\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}}\left(\frac{\left[\sum_{m=0}^{2}s_{lm}F_m\left(x,0;\Sigma,\Xi\right)\right]}{\prod_{i\neq l}\left(\sum_{k=0}^{2}\left(s_{lk}-s_{ik}\right)F_k\left(x,0;\Sigma,\Xi\right)\right)}\right.\\ & \left.\left. + \frac{8}{\mathcal{R}}\zeta^2\,\frac{\partial}{\partial y}\frac{\left[\sum_{m=0}^{2}s_{lm}F_m\left(x,y;\Sigma,\Xi\right)\right]}{\prod_{i\neq l}\left(\sum_{k=0}^{2}\left(s_{lk}-s_{ik}\right)F_k\left(x,y;\Sigma,\Xi\right)\right)}\right|_{y=0} + \cdots\right)\right\}. \end{aligned} \tag{D.17}$$

Two different terms, $I_2^A(r) + I_2^B(r) = I_2(r)$, can be identified in (D.17). The evaluation of the first term is straightforward:

$$\begin{aligned} I_2^A(r) \sim\ & \frac{\Xi D_0 L^2}{2}\int_0^{\infty}\frac{d\zeta}{(2\pi)^2}\zeta^s e^{-\zeta^2}\int_{-1}^{1}dx\left(e^{i\zeta\frac{r}{L}x}-1\right)\left(1+\frac{8}{\mathcal{R}}\zeta^2+\cdots\right)\\ & = \frac{\Xi D_0}{2(2\pi)^2}r^2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\Gamma(2n+4)}\left(\Gamma\left(\frac{s+2+2n}{2}\right)+\frac{8}{\mathcal{R}}\Gamma\left(\frac{s+5+2n}{2}\right)\right)\\ & \times\left(\frac{r}{L}\right)^{2n}\left(1+\frac{4(s+2)}{\mathcal{R}}+\cdots\right). \end{aligned} \tag{D.18}$$

The evaluation of the second term is more cumbersome. The leading term can be recasted in the form:

$$I_2^B(r) \sim \frac{8D_0L^2}{\mathcal{R}}\int_0^\infty \frac{d\zeta}{(2\pi)^2}\zeta^s e^{-\zeta^2}\int_{-1}^1 dx\left(e^{i\zeta\frac{r}{L}x}-1\right)\frac{\left(1-x^2\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}}\sum_{n=0}^\infty A_n(\Sigma)x^{2n}. \tag{D.19}$$

The coefficients A_i are Σ-dependent numerical constants. The first two of them are given by the expressions

$$\begin{aligned} A_0(\Sigma) &= \frac{1}{16\sqrt{3}\left(1-\sin\frac{\pi}{6}\right)\cos\left(\frac{1}{3}\tan^{-1}\sqrt{26}\right)},\\ A_1(\Sigma) &= -\frac{65\sin\left(\frac{2}{3}\tan^{-1}\sqrt{26}\right)}{512\sqrt{26}\cos^2\left(\frac{1}{3}\tan^{-1}\sqrt{26}\right)}\Sigma^2,\ldots \end{aligned} \tag{D.20}$$

The exact form of these coefficients is however irrelevant for our analysis. Some tedious standard calculations yield:

$$I_2^B(r) = \frac{D_0\mathcal{R}^{\frac{1}{3}}}{\pi\Gamma\left(\frac{2}{3}\right)}\left(\frac{\nu}{U}\right)^{\frac{4}{3}} r^{\frac{2}{3}}\sum_{n=0}^\infty C_n(\Sigma)\Gamma\left(\frac{3s+3n+5}{6}\right)\left(\frac{r}{L}\right)^n, \tag{D.21}$$

where the coefficients $C_n(\Sigma)$ depend on the constants A_i. For $n=0$ one has

$$C_0(\Sigma)=\frac{54\sqrt{3}-74}{27\sqrt{3}}A_0+\frac{128}{9\sqrt{3}}A_1(\Sigma). \tag{D.22}$$

The comparison between $I_2^B(r)$ and $I_2^A(r)$ indicate that a crossover between the corresponding scaling behaviors occurs at

$$r \sim \left|2\times 8.328\sqrt{\pi}\frac{0.0336-0.1127\cot^2\theta_U}{2+\sin\theta_U\cos\theta_U}\right|^{\frac{3}{4}}\mathcal{R}^{-\frac{3}{4}}L. \tag{D.23}$$

For the perturbative expansion in $\frac{1}{\mathcal{R}}$ to be meaningful, the parameter θ_U must have a value close to $\frac{\pi}{2}$. This implies:

$$r \sim F\mathcal{R}^{-\frac{3}{4}}L, \quad \text{with } F \sim 0.6.$$

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